ON AN AXIALLY SYMMETRICAL CONTACT PROBLEM FOR A NON-PLANE DIE CIRCULAR IN PLAN

(OB ODNOI OSESIMMETRICHNOI KONTAKTNOI ZADACHE DLIA NEPLOSKOGO SHTAMPA KRUGOVOGO V PLANE)

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E. G. DEICH
(Bucharest)

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1. Let us assume that in the initial state the elevation of points of the die surface above the plane which forms the boundary of the elastic half space is given by the equation

$$\varphi(r) = Ar^{2\lambda} \qquad (\lambda > 0) \tag{1.1}$$

Suppose that a force Q is applied along the axis of rotation of the base surface of the die. For the radius of the contact area a and the displacement of the die δ we have [1]

$$a^{2\lambda+1} = \frac{(\nu-1)\,Q}{4\nu GA\beta_\lambda}, \qquad \delta = Aa^{2\lambda}\alpha_\lambda \tag{1.2}$$

where

$$\alpha_\lambda = \sum_{k=0}^{+\infty} I_k^{(\lambda)} \frac{(-1)^k 2^k k!}{(2k-1)!!} \qquad [(-1)!! = 1], \qquad \beta_\lambda = \alpha_\lambda - I_0^{(\lambda)} \tag{1.3}$$

$$I_k^{(\lambda)} = (-1)^k (4k+1) \frac{(2k-1)!!}{2^{k+1} k!} \sqrt{\pi}\, \frac{\Gamma^2(\lambda+1)}{\Gamma(k+\lambda+{}^3/_2)\,\Gamma(\lambda-k+1)} \tag{1.4}$$

Here ν is Poisson's ratio, G is the shear modulus of the elastic half space.

For the distribution of pressure over the contact area we have

$$p(r) = \frac{p^\circ}{2\beta_\lambda \mu} \sum_{k=1}^{+\infty} \frac{(-1)^k 2^k k!\, I_k^{(\lambda)}}{(2k-1)!!} \left[1 - \frac{P_{2k}(\mu)}{P_{2k}(0)}\right] \tag{1.5}$$

$$\mu = \sqrt{1 - \frac{r^2}{a^2}} \qquad \left(p^\circ = \frac{Q}{\pi a^2}\right)$$

Here $P_{2k}(\mu)$ are Legendre polynomials. The pressure at the center of the contact area is equal to

$$p_0 = p(0) = p^\circ \chi_\lambda, \qquad \chi_\lambda = \frac{1}{2} - \frac{1}{2\beta_\lambda} \sum_{k=1}^{+\infty} \frac{2^{2k}(k!)^2}{[(2k-1)!!]^2} I_k^{(\lambda)} \tag{1.6}$$

Below, we attempt to put these formulas into a simpler form.

2. Let us introduce the designation

$$\gamma_\lambda = \sum_{k=0}^{+\infty} \frac{2^{2k}(k!)^2}{[(2k-1)!!]^2} I_k^{(\lambda)} \tag{2.1}$$

For the value of χ_λ we have from (1.6)

$$\chi_\lambda = \frac{1}{2} - \frac{1}{2\beta_\lambda}(\gamma_\lambda - I_0^{(\lambda)}) = \frac{\beta_\lambda - \gamma_\lambda + I_0^{(\lambda)}}{2\beta_\lambda} \qquad \text{or} \qquad \chi_\lambda = \frac{\alpha_\lambda - \gamma_\lambda}{2\beta_\lambda} \tag{2.2}$$

Putting the expression for $I_k^{(\lambda)}$ from (1.4) into the form

$$I_k^{(\lambda)} = \frac{(-1)^k (4k+1)}{2k!} \frac{\Gamma(k+{}^1/_2)\,\Gamma^2(\lambda+1)}{\Gamma(k+\lambda+{}^3/_2)\,\Gamma(\lambda-k+1)} \tag{2.3}$$

One can write the Formulas (1.3) and (2.1) as follows:

$$\alpha_\lambda = \frac{\sqrt{\pi}\,\Gamma^2(\lambda+1)}{2} \sum_{k=0}^{+\infty} \frac{4k+1}{\Gamma(k+\lambda+{}^3/_2)\,\Gamma(\lambda-k+1)} \tag{2.4}$$

$$\gamma_\lambda = \frac{\pi\Gamma^2(\lambda+1)}{2} \sum_{k=0}^{+\infty} \frac{(-1)^k (4k+1)\,\Gamma(k+1)}{\Gamma(k+{}^1/_2)\,\Gamma(k+\lambda+{}^3/_2)\,\Gamma(\lambda-k+1)} \tag{2.5}$$

Observing that

$$\frac{1}{\Gamma(\lambda-k+1)} = (-1)^{k+1} \frac{\sin\lambda\pi}{\pi}\,\Gamma(k-\lambda), \qquad 4k+1 = 4\frac{\Gamma(k+{}^5/_4)}{\Gamma(k+{}^1/_4)}$$

the preceding formulas after some transformations can be brought into the following form

$$\alpha_\lambda = \frac{\sqrt{\pi}\,\Gamma(\lambda+1)}{2\Gamma(\lambda+{}^3/_2)}\, {}_3F_2\left(-\lambda,\ \frac{5}{4},\ 1;\ \lambda+\frac{3}{2},\ \frac{1}{4};\ -1\right) \tag{2.6}$$

$$\gamma_\lambda = \frac{\sqrt{\pi}\,\Gamma(\lambda+1)}{2\Gamma(\lambda+{}^3/_2)}\, {}_4F_3\left(-\lambda,\ \frac{5}{4},\ 1,\ 1;\ \lambda+\frac{3}{2},\ \frac{1}{4},\ \frac{1}{2};\ 1\right) \tag{2.7}$$

where ${}_pF_q(a_1, a_2, \ldots, a_p;\ b_1, b_2, \ldots, b_q;\ x)$ is the generalized hypergeometric function [2].

Let us utilize the Whipple's formula [3]

$$\begin{aligned} &{}_4F_3\left(\alpha,\ 1+\frac{1}{2}\alpha,\ \beta,\ \gamma;\ \frac{1}{2}\alpha,\ \alpha-\beta+1,\ \alpha-\gamma+1;\ -1\right) = \\ &= \frac{\Gamma(\alpha-\beta+1)\,\Gamma(\alpha-\gamma+1)}{\Gamma(\alpha+1)\,\Gamma(\alpha-\beta-\gamma+1)} \qquad (\alpha-2\beta-2\gamma > -2) \end{aligned}$$

and the Dougall's formula [3]

$$
{}_5\Gamma_4\left(\alpha,\ 1+\frac{1}{2}\alpha,\ \beta,\ \gamma,\ \delta;\ \frac{1}{2}\alpha,\ \alpha-\beta+1,\ \alpha-\gamma+1,\ \alpha-\delta+1;\ 1\right)=
$$

$$
=\frac{\Gamma(\alpha-\beta+1)\,\Gamma(\alpha-\gamma+1)\,\Gamma(\alpha-\delta+1)\,\Gamma(\alpha-\beta-\gamma-\delta+1)}{\Gamma(\alpha+1)\,\Gamma(\alpha-\beta-\gamma+1)\,\Gamma(\alpha-\gamma-\delta+1)\,\Gamma(\alpha-\delta-\beta+1)}\quad(\alpha-\beta-\gamma-\delta>-1)
$$

Setting in these formulas $\alpha = 1/2$, $\beta = -\lambda$, $\gamma = \delta = 1$, we obtain

$$
\begin{aligned}
&{}_3F_2\left(-\lambda,\ \frac{5}{4},\ 1;\ \lambda+\frac{3}{2},\ \frac{1}{4};\ -1\right)=2\lambda+1 && \left(\lambda>-\frac{1}{4}\right)\\
&{}_4F_3\left(-\lambda,\ \frac{5}{4},\ 1,\ 1;\ \lambda+\frac{3}{2},\ \frac{1}{4},\ \frac{1}{2};\ 1\right)=-\frac{2\lambda+1}{2\lambda-1} && \left(\lambda>\frac{1}{2}\right)
\end{aligned}
\tag{2.8}
$$

Thus, the Formulas (2.6) and (2.7) will, respectively, take the form

$$
\alpha_\lambda=\frac{\sqrt{\pi}\,\Gamma(\lambda+1)}{\Gamma(\lambda+{}^1/_2)},\qquad \gamma_\lambda=-\frac{\sqrt{\pi}\,\Gamma(\lambda+1)}{(2\lambda-1)\,\Gamma(\lambda+{}^1/_2)}\qquad\left(\lambda>\frac{1}{2}\right)\tag{2.9}
$$

Then from (1.3) and (2.2), respectively, we have

$$
\beta_\lambda=\frac{\lambda\sqrt{\pi}\,\Gamma(\lambda+1)}{\Gamma(\lambda+{}^3/_2)},\qquad \chi_\lambda=\frac{2\lambda+1}{2(2\lambda-1)}\qquad\left(\lambda>\frac{1}{2}\right)\tag{2.10}
$$

The Formulas (1.2) and (1.6) then become

$$
a^{2\lambda+1}=\frac{(\nu-1)\,Q}{4\nu GA}\,\frac{\Gamma(\lambda+{}^3/_2)}{\lambda\sqrt{\pi}\,\Gamma(\lambda+1)},\qquad \delta=Aa^{2\lambda}\,\frac{\sqrt{\pi}\,\Gamma(\lambda+1)}{\Gamma(\lambda+{}^1/_2)}\tag{2.11}
$$

$$
p_0=\frac{2\lambda+1}{2(2\lambda-1)}\,p^\circ\qquad\left(\lambda>\frac{1}{2}\right)\tag{2.12}
$$

The Expression (2.10) for β_λ also simplifies the expression ([1], ch. 5, Formula (6.16)) for the potential.

It is difficult to obtain a simpler formula for the pressure using the same method. To that end we choose another way.

3. It is known [4], that the problem under consideration can also be solved in a different manner, namely: the quantities a and δ are given by the formulas

$$
a^2\int_0^{\pi/2}\varphi'(a\sin\theta)\sin^2\theta\,d\theta=\frac{(\nu-1)\,Q}{4\nu G},\qquad \delta=a\int_0^{\pi/2}\varphi'(a\sin\theta)\,d\theta\tag{3.1}
$$

and the pressure p - by the formula

$$
p(r)=-\frac{1}{2\pi}\int_r^a\frac{F'(s)\,ds}{\sqrt{s^2-r^2}}\qquad(0<r<a)\tag{3.2}
$$

where

$$F(r)=\frac{4\nu G}{\nu-1}\left[\delta-r\int_0^{\pi/2}\varphi'(r\sin\theta)\,d\theta\right]\qquad(0\leqslant r\leqslant a)\tag{3.3}$$

Replacing (1.1) by the Expressions (3.1) we readily obtain the Formulas (2.11). Bearing in mind (2.11), from (3.3) we get

$$F(r)=\frac{(2\lambda+1)\,Q}{2\lambda a^{2\lambda+1}}\,(a^{2\lambda}-r^{2\lambda})\qquad(0\leqslant r\leqslant a)\tag{3.4}$$

Now, from (3.2) we find

$$p(r)=\frac{(2\lambda+1)\,Q}{2\pi a^{2\lambda+1}}\int_r^a\frac{s^{2\lambda-1}\,ds}{\sqrt{s^2-r^2}}\tag{3.5}$$

The substitution $s^2 - r^2 = a^2\xi^2$ transforms it into

$$p(r)=\left(\lambda+\frac{1}{2}\right)p^\circ\int_0^\rho\left(\frac{r^2}{a^2}+\xi^2\right)^{\lambda-1}d\xi\qquad\left(\begin{array}{c}0\leqslant r\leqslant a\\ \rho=\sqrt{1-r^2/a^2}\end{array}\right)\tag{3.6}$$

Assuming that $r \neq 0$ and substituting $s^2 = r^2/(1 - t)$ into (3.5) we get

$$p(r)=\frac{2\lambda+1}{4}\,p^\circ\left(\frac{r}{a}\right)^{2\lambda-1}B_{\rho^2}\left(\frac{1}{2},\ \frac{1}{2}-\lambda\right)\qquad\left(\begin{array}{c}0< r\leqslant a\\ \rho^2=1-r^2/a^2\end{array}\right)\tag{3.7}$$

where $B_x(\alpha, \beta)$ is the incomplete beta-function.

Taking into consideration the equalities

$$B_x(\alpha,\ \beta)=\int_0^x t^{\alpha-1}(1-t)^{\beta-1}\,dt=\frac{x^\alpha}{\alpha}\,F(1-\beta,\ \alpha;\ \alpha+1;\ x)\tag{3.8}$$

the equality (3.7) can also be written in the form

$$p(r)=\left(\lambda+\frac{1}{2}\right)p^\circ\left(\frac{r}{a}\right)^{2\lambda-1}\sqrt{1-\frac{r^2}{a^2}}\,F\left(\lambda+\frac{1}{2},\ \frac{1}{2};\ \frac{3}{2};\ 1-\frac{r^2}{a^2}\right)\quad(0< r\leqslant a)\tag{3.9}$$

For the pressure p_0 which corresponds to $r = 0$, from (3.5) or (3.6) we get (2.12).

4. Let us consider some particular cases.

a) Let $\lambda = n$, where n is an integer. In this case we see from (3.6) that

$$p(r)=\frac{2n+1}{2(2n-1)}\,p^\circ\sqrt{1-\frac{r^2}{a^2}}\,E_{n-1}\left(\frac{r^2}{a^2}\right)\qquad(0\leqslant r\leqslant a)\tag{4.1}$$

Here $E_{n-1}(x)$ is a polynomial of the $(n - 1)$th degree with respect to x, defined by the relations

$$E_n(x) = \frac{2n+1}{\sqrt{1-x}} \int_0^{\sqrt{1-x}} (x+\xi^2)^n \, d\xi, \qquad E_n(x) = 1 + \frac{2n}{2n-1} x E_{n-1}(x) \tag{4.2}$$

The recurrence formula here is obtained by integrating the first equality by parts.

The recurrence formula is also correct for the case when n is not an integer. From (4.2) we find $E_0(x) = 1$, and then obtain consecutively from (4.2)

$$E_1(x) = 1 + 2x, \qquad E_3(x) = \frac{1}{5}(5 + 6x + 8x^2 + 16x^3) \tag{4.3}$$

$$E_2(x) = \frac{1}{3}(3 + 4x + 8x^2), \qquad E_4(x) = \frac{1}{35}(35 + 40x + 48x^2 + 64x^3 + 128x^4)$$

It is easy to verify that polynomials $E_n(x)$ are the Jacobi polynomials, namely $E_n(x) = F(-n,\ 1;\ 1/2 - n;\ x)$. Thus, the Expression (4.1) can also be presented in the form

$$p(r) = \frac{2n+1}{2(2n-1)} p^\circ \sqrt{1 - \frac{r^2}{a^2}} F\left(1-n,\ 1;\ \frac{3}{2} - n;\ \frac{r^2}{a^2}\right) \qquad (0 \leqslant r \leqslant a) \tag{4.4}$$

For instance, from (4.4) and (2.11) with $n = 1$ and $A = 1/2R$ we get

$$p(r) = \frac{3}{2} p^\circ \sqrt{1 - \frac{r^2}{a^2}} \qquad (0 \leqslant r \leqslant a), \qquad a^3 = \frac{3(\nu - 1) QR}{8\nu G}, \qquad \delta = \frac{a^2}{R}$$

b) Let $\lambda = 1/2$, $A = \cot\alpha$, i.e. the case of the pointed end of a conical die being pressed into the elastic half-space. From (2.11) we find

$$a^2 = \frac{(\nu - 1) Q \tan\alpha}{\pi \nu G}, \qquad \delta = \frac{\pi}{2} a \cot\alpha$$

and from (3.9)

$$p(r) = p^\circ \sqrt{1 - \frac{r^2}{a^2}} F\left(1,\ \frac{1}{2};\ \frac{3}{2};\ 1 - \frac{r^2}{a^2}\right) = \frac{1}{2} p^\circ \ln \frac{a + \sqrt{a^2 - r^2}}{a - \sqrt{a^2 - r^2}} \qquad (0 < r \leqslant a) \tag{4.5}$$

Here we have used the relation [2]

$$F\left(1,\ \frac{1}{2};\ \frac{3}{2};\ x^2\right) = \frac{1}{2x} \ln \frac{1+x}{1-x}$$

c) Let $\lambda = 3/2$. From (2.11) we have

$$a^4 = \frac{4(\nu - 1) Q}{9\pi \nu G A}, \qquad \delta = \frac{3\pi}{4} A a^3$$

From (3.6) we obtain

$$p(r) = p^\circ \left[\sqrt{1 - \frac{r^2}{a^2}} + \frac{r^2}{a^2} \ln \frac{a + \sqrt{a^2 - r^2}}{r} \right] \qquad (0 \leqslant r \leqslant a)$$

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